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Exact solution of an inverse problem for affine images of IFS attractors

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Abstract

A contractive iterated function system (IFS) defined on a complete metric space X possesses a unique compact attractor $F \subset X$. Obtaining the transformations of F is generally nontrivial, due to the nature of construction of F . In this work, we review a method based on barycentric coordinates for constructing affine transformations of F (translation, scaling, reflection, rotation, shear, and their compositions) and develop its stronger theoretical framework. This approach yields an exact solution to a specific inverse problem: given the affine image $w(F)$ as a target set, we derive an explicit formula for the IFS whose fixed point is exactly $w(F)$. Taking into account the definition of the Hutchinson operator, we also obtain the corresponding Hutchinson operator, whose fixed point is also $w(F)$. Pseudocode and examples are included to facilitate implementation and verification.

Keywords: Inverse problem; Iterated function system; Hutchinson operator; Affine transformation; Barycentric coordinates; Fractal attractor

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1 Introduction

An iterated function system (IFS) is a finite set of mappings, usually contractive, which, when applied iteratively to a compact set, produce an invariant set. IFSs are widely recognized as a theoretical framework for defining, analyzing, and generating fractals. They also have numerous applications in physics, mathematics, biology, engineering, computer science, and computer graphics [17]. Hutchinson [22] first employed IFS as a tool for constructing fractals, although not under this name. The term ‘iterated function system’ was later introduced by Barnsley and Demko [5]. Subsequent works [3, 14, 28] further developed the theory and investigated its applications in greater depth.

In this section we first provide precise definitions of an IFS and Hutchinson operator, along with key theorems relevant to our work. We then state the inverse problem that is the main focus of our paper. In the next section we present fractal transformations as well as definitions and essential facts about barycentric coordinates, and also the method of affine transformation of fractals that uses barycentric coordinates. The main section comes next, it contains the exact solution of the stated inverse problem. The Examples section illustrates our method in practice. The last section contains concluding remarks.

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1.1 Iterated function system and Hutchinson operator

Let (X, d) be a complete metric space.

Definition 1.1 The mapping $w : X \rightarrow X$ is called *contractive* if there exists a contractivity factor $c \in [0, 1)$ such that

$$d(w(\mathbf{x}), w(\mathbf{y})) \leq c \cdot d(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in X. \quad (1.1)$$

Definition 1.2 A *contractive iterated function system* consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_i : X \rightarrow X$ with respective contractivity factors s_i for $i = 1, 2, \dots, N$, where $N \in \mathbb{N}$. Its *contractivity factor* is $s = \max\{s_i : i = 1, 2, \dots, N\}$, $s < 1$, while its notation is $S = \{X; w_i, i = 1, 2, \dots, N\}$.

Let $(\mathcal{H}(X), h(d))$ denote the space of nonempty compact subsets of X , equipped with the Hausdorff metric $h(d)$ induced by d , which is defined for any $A, B \in \mathcal{H}(X)$ by

$$h(d)(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right\}.$$

Definition 1.3 Let $S = \{X; w_i, i = 1, 2, \dots, N\}$ be a contractive iterated function system with a contractivity factor s . Then the operator W_S defined on $(\mathcal{H}(X), h(d))$ by

$$W_S(B) = \cup_{i=1}^N w_i(B), \quad B \in \mathcal{H}(X),$$

is called *Hutchinson operator* (associated to the IFS S).

Theorem 1.4 ([3]) *The Hutchinson operator W_S is a contraction mapping on the complete metric space $(\mathcal{H}(X), h(d))$ with contractivity factor s . Its unique fixed point, $F \in \mathcal{H}(X)$ obeys*

$$F = W_S(F) = \cup_{i=1}^N w_i(F),$$

and is given by $F = \lim_{n \rightarrow \infty} W_S^{\circ n}(B)$, for any $B \in \mathcal{H}(X)$ (here, $W_S^{\circ n}$ denotes the n -th forward iterate of W_S).

The fixed point of the Hutchinson operator W_S , also called the *attractor of the IFS S* , $\text{att}(S)$ might have smooth or fractal shape ([24]). As smooth forms can be defined in much simpler ways, usually the researchers are interested in fractal attractors of an IFS.

The two mostly used algorithms for generating the attractor of an IFS are the deterministic and the random iteration algorithm. The deterministic approach follows directly from Theorem 1.4. Starting with an arbitrary compact set $F_0 \subset X$, one iteratively applies the Hutchinson operator,

$$F_{n+1} = W_S(F_n), \quad n = 0, 1, 2, \dots$$

so that the sequence $\{F_n\}$ converges to the attractor, i.e.

$$F = \lim_{n \rightarrow \infty} F_n = \lim_{n \rightarrow \infty} W_S^{\circ n}(F_0).$$

In contrast, the random iteration algorithm (RIA) provides a stochastic approach to generating the same attractor. Starting from an arbitrary initial point $x_0 \in X$, one chooses at each step a mapping $w_i \in W_S$ according to a given probability distribution (w_i is chosen with probability p_i , s.t. $p_1 + p_2 + \dots + p_N = 1$). In such a way, a sequence is obtained,

$$x_{n+1} = w_i(x_n), \quad n = 0, 1, 2, \dots,$$

which, under certain conditions, converges with probability one [3, 9] to the attractor F . In practice, after a sufficient number of iterations, the points x_n approximate the attractor and can be visualized to reveal its structure. In the next sections of this paper, we will employ the latter approach, taking $X = \mathbb{R}^m$ and $d = d_E$, where d_E stands for Euclidean metric, to enable a clearer and more intuitive depiction of fractals. In the Examples section we work with $X = \mathbb{R}^2$.

1.2 Inverse problem

We have just described the direct problem: how to determine the attractor of a given IFS S .

The focus of this paper, however, is on the corresponding inverse problem. In general, the inverse problem can be formulated as follows: given a target element T in a complete metric space (X, d) , find an appropriate contraction mapping (Hutchinson operator or IFS) whose fixed point *approximates* T .

The famous Collage Theorem (given below), formulated by Barnsley lies at the core of most of the inverse methods. It enables finding attractor close to the target set, by minimizing the so-called collage distances. This theorem gives not only theoretical guarantees, but also a practical method for constructing the approximation of the target set. In some simple cases, the IFS that generates the approximation can be done even manually.

Theorem 1.5 (*Collage Theorem, [6]*) *Let (X, d) be a metric space. Suppose T is a nonempty, compact subset of X and let $\varepsilon > 0$ be given. Choose an iterated function system (IFS) $S = \{X; w_1, w_2, \dots, w_N\}$ with contractivity factor s , $0 \leq s < 1$. Suppose $h(d)(T, \bigcup_{n=1}^N w_n(T)) \leq \varepsilon$, where $h(d)$ is the Hausdorff metric. Then*

$$h(d)(T, A) \leq \frac{\varepsilon}{1-s},$$

where A is the attractor of the IFS S .

This theorem became strongly associated to fractal image coding, i.e. fractal image compression. Representative papers on the usage of the Collage Theorem in solving the inverse problem were written by Barnsley and his collaborators [2, 6]. Jacquin [23] has introduced the Partitioned Iterated Function Systems, which provided an efficient scheme for fractal image coding by partitioning the image and searching for contractive mappings between the parts. Some later works focus on algorithmic implementation of this theorem for binary images [15], while others employ machine learning to estimate contractive affine transformations directly from data [18].

In this work, we address a specific instance of the inverse problem: given an IFS with attractor F and an affine transformation w (translation, scaling, reflection, rotation, shear, or any finite composition), the goal is to construct an IFS whose attractor is exactly $w(F)$. The solution formulated in Theorem 3.2 shows the advantage of our approach: it provides an exact solution rather than an approximation.

2 Barycentric coordinates as a tool for affine transformation of fractals

2.1 Fractal transformations

Since the fractal representations of natural objects were shown to be very authentic [3, 27], the scientific community has been interested in modeling fractals in purpose of better representation of objects. Common transformations of fractals include affine [1, 3, 10, 19, 21, 25, 26], Möbius, and analytic transformations [3], which affect the entire attractor. Transformations have also been defined between two attractors, where a mapping sends one precisely onto the other [4, 7, 8].

Other methods act locally by adjusting selected coefficients of the IFS code [11, 12]. The theorem established in [3], p. 113, states that the image of an IFS fractal depends continuously on the coefficients of the affine mappings in the IFS code. Importantly, while these changes are continuous, they are not necessarily predictable. In some cases, attractors exhibit highly complex, non-predictable behaviour, even when a single coefficient of one of the IFS mappings changes by about 10^{-1} . In the same book, Barnsley also introduced another transformation approach—directly transforming each fractal point—which, however, appears impractical for real-time interactive fractal manipulation. The barycentric coordinates were used for the first time by Kocic and his collaborators [25, 26] in forming the so called affinely invariant iterated function system, which enables affine transformation of fractals. While in this approach the user is only aware of the original fractal and the final transformation, there are software applications created by the author of this paper and collaborators [19–21, 29] that work in real time and the transformations are done interactively, by clicking and dragging predefined points (points that form an affine basis).

In the following subsections, we build upon this last approach, adding a stronger theoretical framework that will lead to the exact solution of the inverse problem formulated in Sect. 1.2.

2.2 Barycentric coordinates - definitions and basic properties

The following definitions from barycentric calculus are drawn primarily from [30], with additional references from [16] and [13].

Definition 2.1 A set $\mathcal{A}$ of k points, $\mathcal{A} = \{A_1, A_2, \dots, A_k\} \subset \mathbb{R}^m$, $m \geq 2$, is called *pointwise independent set*, if the $k-1$ vectors $\overrightarrow{A_1 A_j}$, $j = 2, 3, \dots, k$, are linearly independent. The points $A_1, A_2, \dots, A_k$ are called *pointwise independent*. The set of $m+1$ pointwise independent points from $\mathbb{R}^m$, is called *affine basis of $\mathbb{R}^m$* .

Definition 2.2 Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a pointwise independent set of points from $\mathbb{R}^m$. Then, the real numbers $a_1, a_2, \dots, a_k$ are called *barycentric (or affine) coordinates of the point $P \in \mathbb{R}^m$ with respect to the set $\mathcal{A}$* , if

$$P = \sum_{i=1}^k a_i A_i, \quad \sum_{i=1}^k a_i = 1. \quad (2.1)$$

We will use the notation $(a_1, a_2, \dots, a_k)_{\mathcal{A}}$ or $\mathbf{p}_{\mathcal{A}}$.

Regarding the last two definitions, we note that:

1. $a_1, a_2, \dots, a_k$ are barycentric coordinates of the point P with respect to the set $\{A_1, A_2, \dots, A_k\}$ if and only if for any $i \in \{1, 2, \dots, k\}$ the following vector equation holds:

$$\overrightarrow{A_i P} = a_1 \overrightarrow{A_i A_1} + \dots + a_{i-1} \overrightarrow{A_i A_{i-1}} + a_{i+1} \overrightarrow{A_i A_{i+1}} + \dots + a_n \overrightarrow{A_i A_k}. \quad (2.2)$$

2. The largest number of affinely independent points in $\mathbb{R}^m$ is $m + 1$ (since the largest number of linearly independent vectors in $\mathbb{R}^m$ is m).
3. When $\mathcal{A} = \{A_1, A_2, \dots, A_{m+1}\}$ is an affine basis of $\mathbb{R}^m$, we will often use two equivalent forms of (2.1). Let $\mathbf{p} = [p_1 \ p_2 \ \dots \ p_m]^T$ and $\mathbf{A}_i = [a_{i1} \ a_{i2} \ \dots \ a_{im}]^T$ be vectors presenting Cartesian (rectangular) coordinates of the points P and A_i , $i = 1, 2, \dots, m + 1$, respectively, and $\mathbf{p}_{\mathcal{A}} = [a_1 \ a_2 \ \dots \ a_{m+1}]^T_{\mathcal{A}}$ be a vector representing barycentric coordinates of the point P with respect to the basis $\mathcal{A}$. Then the three equations (2.1), (2.3) and (2.4) are equivalent:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} = \sum_{i=1}^{m+1} a_i \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{im} \end{bmatrix}, \quad \sum_{i=1}^{m+1} a_i = 1 \quad (2.3)$$

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \mathbf{A} \cdot \mathbf{p}_{\mathcal{A}}, \quad \text{where } \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_{m+1} \\ 1 & 1 & \dots & 1 \end{bmatrix}. \quad (2.4)$$

4. The matrix

$$\begin{aligned} \mathbf{M} &= [\mathbf{A}_1 \ \mathbf{A}_2 \ \dots \ \mathbf{A}_m] + \mathbf{A}_{m+1} \cdot (-\mathbf{1}^T) \\ &= \begin{bmatrix} a_{11} - a_{1,m+1} & a_{12} - a_{1,m+1} & \dots & a_{1m} - a_{1,m+1} \\ a_{21} - a_{2,m+1} & a_{22} - a_{2,m+1} & \dots & a_{2m} - a_{2,m+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} - a_{m,m+1} & a_{12} - a_{m,m+1} & \dots & a_{mm} - a_{m,m+1} \end{bmatrix} \\ &= \begin{bmatrix} \overrightarrow{A_{m+1}A_1} & \overrightarrow{A_{m+1}A_2} & \dots & \overrightarrow{A_{m+1}A_m} \end{bmatrix} \end{aligned}$$

is invertible, since the vectors $\overrightarrow{A_{m+1}A_1}, \overrightarrow{A_{m+1}A_2}, \dots, \overrightarrow{A_{m+1}A_m}$ are linearly independent. Also, $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \left[\begin{array}{c|c} \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{A}_{m+1} \\ \hline (-\mathbf{1}^T)\mathbf{M} & 1 + \mathbf{1}^T\mathbf{M}^{-1}\mathbf{A}_{m+1} \end{array} \right]. \quad (2.5)$$

5. The relation (2.4) presents the *conversion* from barycentric coordinates with respect to $\mathcal{A} = \{A_1, A_2, \dots, A_{m+1}\}$, $\mathbf{p}_{\mathcal{A}}$, to rectangular coordinates $\mathbf{p}$ of a point P . The opposite conversion, from rectangular to barycentric with respect to $\mathcal{A}$, is given by

$$\mathbf{p}_{\mathcal{A}} = \mathbf{A}^{-1} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}. \quad (2.6)$$

These conversions ensure one-to-one correspondence between rectangular and barycentric coordinates with respect to an affine basis, of a point $P \in \mathbb{R}^m$.

2.3 Affine transformations of fractals using barycentric coordinates

Let the affine transformation w be given by

$$w: \mathbf{x} \mapsto \mathbf{B}\mathbf{x} + \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}^m. \quad (2.7)$$

Affine transformations keep affine combinations. Namely, for given set of points $\{A_1, A_2, \dots, A_k\}$ and for any $a_1, a_2, \dots, a_k \in \mathbb{R}$, such that $a_1 + a_2 + \dots + a_k = 1$, the next equality follows directly from the definition of affine transformation:

$$w(a_1 A_1 + a_2 A_2 + \dots + a_k A_k) = a_1 w(A_1) + a_2 w(A_2) + \dots + a_k w(A_k). \quad (2.8)$$

Suppose that $\mathcal{A} = \{A_1, A_2, \dots, A_{m+1}\}$ is an affine basis of $\mathbb{R}^m$. Given an arbitrary point $P = (a_1, a_2, \dots, a_{m+1})_{\mathcal{A}}$, $a_1 + a_2 + \dots + a_{m+1} = 1$. Then, according to (2.8),

$$w((a_1, a_2, \dots, a_{m+1})_{\mathcal{A}}) = (a_1, a_2, \dots, a_{m+1})_{w(\mathcal{A})}. \quad (2.9)$$

The last equality means that $w(P)$ has the same relative position w.r.t. the basis $w(\mathcal{A})$, as P has w.r.t. $\mathcal{A}$. This claim holds in the opposite direction as well, formulated as Lemma 2.3.

Lemma 2.3 *Let w maps the pointwise independent points $A_1, A_2, \dots, A_k$ into the pointwise independent points $A'_1, A'_2, \dots, A'_k$. If the point P' with respect to $A'_1, A'_2, \dots, A'_k$ and the point P with respect to $A_1, A_2, \dots, A_k$ have the same barycentric coordinates $(a_1, a_2, \dots, a_k)$, then*

$$P' = w(P). \quad (2.10)$$

Proof

$$\begin{aligned} P' &= (a_1, a_2, \dots, a_k)_{\mathcal{A}'} \\ &= a_1 A'_1 + a_2 A'_2 + \dots + a_k A'_k \\ &= a_1 w(A_1) + a_2 w(A_2) + \dots + a_k w(A_k) \\ &= w(a_1 A_1 + a_2 A_2 + \dots + a_k A_k) \\ &= w((a_1, a_2, \dots, a_k)_{\mathcal{A}}) = w(P). \end{aligned} \quad \square$$

3 Exact construction of an IFS for a given affine image of an attractor

Based on the theoretical framework introduced in Sect. 2.2, we can formulate and prove the following two theorems.

Theorem 3.1 *Let $w(\mathbf{x}) = \mathbf{B}\mathbf{x} + \mathbf{b}$ be an affine transformation defined on $\mathbb{R}^m$ and let $\mathcal{A} = \{A_1, A_2, \dots, A_{m+1}\}$ be any affine basis of $\mathbb{R}^m$. Then, with respect to this basis, w is represented by*

$$w(\mathbf{p}_{\mathcal{A}}) = \mathbf{A}^{-1} \left[\begin{array}{c|c} \mathbf{B} & \mathbf{b} \\ \hline \mathbf{0}^T & 1 \end{array} \right] \mathbf{A} \mathbf{p}_{\mathcal{A}}. \quad (3.1)$$

where $\mathbf{p}_A = (a_1, a_2, \dots, a_{m+1})$ are barycentric coordinates of the arbitrary point $\mathbf{p} \in \mathbb{R}^m$ with respect to $\{A_1, A_2, \dots, A_{m+1}\}$ and $\mathbf{A}$ is as defined in (2.4).

Proof Let $\mathbf{p} \in \mathbb{R}^m$, $w(\mathbf{p}) = \mathbf{p}' = (p'_1, p'_2, \dots, p'_m)$ and let the barycentric coordinates of $\mathbf{p}'$ with respect to $\mathcal{A}$ be $\mathbf{p}'_A = (a'_1, a'_2, \dots, a'_{m+1})_A$. Thus, from (2.6), we have

$$\begin{aligned} \mathbf{p}'_A &= \mathbf{A}^{-1} \begin{bmatrix} \mathbf{p}' \\ 1 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \mathbf{B}\mathbf{p} + \mathbf{b} \\ 1 \end{bmatrix} \\ &= \mathbf{A}^{-1} \left(\left[\begin{array}{c|c} \mathbf{B} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{array} \right] \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \right) \\ &= \mathbf{A}^{-1} \left[\begin{array}{c|c} \mathbf{B} & \mathbf{b} \\ \mathbf{0}^T & 1 \end{array} \right] \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \\ &= \mathbf{A}^{-1} \left[\begin{array}{c|c} \mathbf{B} & \mathbf{b} \\ \mathbf{0}^T & 1 \end{array} \right] \mathbf{A} \mathbf{p}_A. \quad \square \end{aligned}$$

Equation (3.1) shows that the affine transformation (2.7) can be expressed as a linear operator on barycentric coordinates. Therefore, we use $\mathbf{L}$ to denote the matrix that represents this linear transformation,

$$\mathbf{L} = \mathbf{A}^{-1} \left[\begin{array}{c|c} \mathbf{B} & \mathbf{b} \\ \mathbf{0}^T & 1 \end{array} \right] \mathbf{A}. \quad (3.2)$$

Affine images of IFS attractors are themselves IFS attractors, as shown in the next theorem.

Theorem 3.2 Let $S = \{\mathbb{R}^m; w_1, w_2, \dots, w_N\}$ be an IFS with $w_i(\mathbf{x}) = \mathbf{B}_i \mathbf{x} + \mathbf{b}_i$, and attractor $F = \text{att}(S)$. Let $w(\mathbf{x}) = \mathbf{B} \mathbf{x} + \mathbf{b}$ be an affine transformation on $\mathbb{R}^m$. Then the affine image $w(F)$ is the attractor of another IFS, $w(F) = \text{att}(S')$, $S' = \{\mathbb{R}^m; w'_1, w'_2, \dots, w'_N\}$ where each $w'_i(\mathbf{x}) = \mathbf{B}'_i \cdot \mathbf{x} + \mathbf{b}'_i$ is given by

$$\left[\begin{array}{c|c} \mathbf{B}'_i & \mathbf{b}'_i \\ \mathbf{0}^T & 1 \end{array} \right] = \mathbf{A}' \mathbf{L}_i (\mathbf{A}')^{-1}, \quad (3.3)$$

with

$$\mathbf{L}_i = \mathbf{A}^{-1} \left[\begin{array}{c|c} \mathbf{B}_i & \mathbf{b}_i \\ \mathbf{0}^T & 1 \end{array} \right] \mathbf{A}.$$

Here $\mathbf{A}$ and $\mathbf{A}'$ are the matrices containing the rectangular coordinates of an arbitrary affine basis' points, A_i , and their images $A'_i = w(A_i)$ as defined in (2.4).

Proof Let $\mathcal{A} = \{A_1, A_2, \dots, A_{m+1}\}$ be an arbitrary affine basis of $\mathbb{R}^m$. The image of this affine basis $\mathcal{A}' = w(\mathcal{A}) = \{A'_1, A'_2, \dots, A'_{m+1}\}$ is also an affine basis of $\mathbb{R}^m$, since affine transformations preserve collinearity. Their matrix representations, according to (2.4), are:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_{m+1} \\ 1 & 1 & \dots & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{bmatrix} \mathbf{A}'_1 & \mathbf{A}'_2 & \dots & \mathbf{A}'_{m+1} \\ 1 & 1 & \dots & 1 \end{bmatrix},$$

where the columns are the corresponding rectangular coordinates of the basis points. Both matrices are invertible (as shown in Sect. 2.2., item 4). According to (3.2), the matrix representations of w_i -s w.r.t. the affine basis $\mathcal{A}$ are given by

$$\mathbf{L}_i = \mathbf{A}^{-1} \left[\begin{array}{c|c} \mathbf{B}_i & \mathbf{b}_i \\ \hline \mathbf{0}^T & 1 \end{array} \right] \mathbf{A}.$$

Let $\mathbf{B}'_i$ and $\mathbf{b}'_i$ be defined by

$$\left[\begin{array}{c|c} \mathbf{B}'_i & \mathbf{b}'_i \\ \hline \mathbf{0}^T & 1 \end{array} \right] = \mathbf{A}' \mathbf{L}_i (\mathbf{A}')^{-1}.$$

The mappings w'_i -s determined by $\mathbf{B}'_i$ and $\mathbf{b}'_i$ are contractive, because $\mathbf{L}_i$ represent the contractive mappings w_i . Thus $S' = \{\mathbb{R}; w'_1, w'_2, \dots, w'_N\}$ possesses an attractor, $C = \text{att}(S')$. We will show that $C = w(F)$, by mathematical induction performed on the attractors points.

Let $F = \{\mathbf{f}_0, \mathbf{f}_1, \dots\}$ and $C = \{\mathbf{c}_0, \mathbf{c}_1, \dots\}$ be obtained by RIA from the IFSs S and S' , respectively, with the same probability distribution and same choice of w_i at every step. We will assume that the initial point of F is an arbitrary $\mathbf{f}_0 \in \mathbb{R}^m$, while the initial point of C is $\mathbf{c}_0 = w(\mathbf{f}_0)$. Let's prove that $\mathbf{c}_{k+1} = w(\mathbf{f}_{k+1})$, based on the fact that $\mathbf{c}_k = w(\mathbf{f}_k)$, for arbitrary $\mathbf{f}_k \in F$.

Let $\mathbf{f}_k = (a_1, a_2, \dots, a_{m+1})_{\mathcal{A}}$. Then, according to (2.9), $\mathbf{c}_k = w(\mathbf{f}_k) = (a_1, a_2, \dots, a_{m+1})_{\mathcal{A}'}$, because $w(\mathcal{A}) = \mathcal{A}'$. For some $i \in \{1, 2, \dots, N\}$,

$$\mathbf{c}_{k+1} = w'_i(\mathbf{c}_k).$$

Therefore, in the basis $\mathcal{A}'$ we have:

$$\begin{aligned} (\mathbf{c}_{k+1})_{\mathcal{A}'} &= (\mathbf{A}')^{-1} \left[\begin{array}{c|c} \mathbf{B}'_i & \mathbf{b}'_i \\ \hline \mathbf{0}^T & 1 \end{array} \right] \mathbf{A}' (\mathbf{c}_k)_{\mathcal{A}'} = (\mathbf{A}')^{-1} \mathbf{A}' \mathbf{L}_i (\mathbf{A}')^{-1} \mathbf{A}' (\mathbf{c}_k)_{\mathcal{A}'} \\ &= \mathbf{L}_i (\mathbf{c}_k)_{\mathcal{A}'} = \mathbf{L}_i (\mathbf{f}_k)_{\mathcal{A}} = (\mathbf{f}_{k+1})_{\mathcal{A}}. \end{aligned}$$

Since the coordinates of $\mathbf{f}_{k+1}$ w.r.t. the affine basis $\mathcal{A}$ are the same as those of $\mathbf{c}_{k+1}$ w.r.t. the basis $\mathcal{A}'$, it follows from Lemma 2.3, we get

$$w(\mathbf{f}_{k+1}) = \mathbf{c}_{k+1}.$$

Hence, by the principal of mathematical induction, we have proved that $w(F) = C = \text{att}(S')$. $\square$

We note that the Hutchinson operator whose fixed point is $w(F)$, is

$$W' = \bigcup_{i=1}^N w'_i, \quad w'_i(\mathbf{x}) = \mathbf{B}'_i \cdot \mathbf{x} + \mathbf{b}'_i.$$

After the Theorem 3.2, it should be clearer why do we employ the random iteration algorithm to represent IFS attractors. Actually, we need the representation of an attractor F as a finite set of points, $F = \{F_1, F_2, \dots, F_n\}$, whose rectangular coordinates can be

Algorithm 1 Constructing the IFS S' , such that $\text{att}(S') = w(F)$

Require: Number of mappings N ; IFS mappings $w_i(\mathbf{x}) = \mathbf{B}_i\mathbf{x} + \mathbf{b}_i$, $i = 1, \dots, N$; affine transformation $w(\mathbf{x}) = \mathbf{B}\mathbf{x} + \mathbf{b}$; affine basis $\{A_1, A_2, \dots, A_{m+1}\}$ in $\mathbb{R}^m$

Ensure: New IFS $S' = \{\mathbb{R}^m; w'_i, i = 1, \dots, N\}$ such that $\text{att}(S') = w(\text{att}(S))$

- 1: Construct the matrix $\mathbf{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_{m+1} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$
 - 2: Compute $A'_i = w(A_i)$ for $i = 1, \dots, m+1$
 - 3: Construct the matrix $\mathbf{A}' = \begin{bmatrix} A'_1 & A'_2 & \cdots & A'_{m+1} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$
 - 4: **for** $i = 1$ to N **do**
 - 5: Compute $\mathbf{L}_i = \mathbf{A}^{-1} \begin{bmatrix} \mathbf{B}_i & \mathbf{b}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{A}$
 - 6: Compute $\begin{bmatrix} \mathbf{B}'_i & \mathbf{b}'_i \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{A}' \mathbf{L}_i (\mathbf{A}')^{-1}$
 - 7: Define $w'_i(\mathbf{x}) = \mathbf{B}'_i\mathbf{x} + \mathbf{b}'_i$
 - 8: **end for**
 - 9: **return** $S' = \{\mathbb{R}^m; w'_1, w'_2, \dots, w'_N\}$
-

converted to barycentric. Choosing an arbitrary affine basis $\mathcal{A} = \{A_1, A_2, \dots, A_{m+1}\}$ and using the equations (2.6) and (2.5), one can compute the barycentric coordinates of F_i , $F_i = (f_{i1}, f_{i2}, \dots, f_{i,m+1})_{\mathcal{A}}$. By only finding the $A'_i = w(A_i)$, $i = 1, 2, \dots, m+1$, and computing the rectangular coordinates of the points $w(F_i) = F'_i = (f'_{i1}, f'_{i2}, \dots, f'_{i,m+1})_{\mathcal{A}'}$ by (2.4), to determine the position of these points, we have the affine image of the attractor F , that is $w(F) = \{F'_1, F'_2, \dots, F'_n\}$. This approach enables real-time interactive affine transformation of fractals, where the transformation is defined and performed by clicking and dragging a basis point A_i , one at a time.

To make our construction more transparent, we provide Algorithm 1, which describes the step-by-step procedure for finding S' , allowing readers both to implement their own code and to verify the correctness of our approach.

We note that the role of the affine basis $\mathcal{A} = \{A_1, A_2, \dots, A_{m+1}\}$ is essential in solving the inverse problem, since S' cannot be determined without the matrices containing the coordinates of A_i and $A'_i = w(A_i)$. The choice of affine basis is completely arbitrary. However, for practical reasons, the so-called standard affine basis—consisting of the coordinate origin O and the points on the coordinate axes at unit distance from O —is often used. This choice provides a clearer geometric interpretation of the affine transformation applied to the fractal, since the same transformation can be simultaneously applied to the points forming the affine basis. In the examples that follow, we will illustrate both standard and nonstandard choices of affine bases.

4 Examples

Example The fractal “Peitgen tree” is the attractor of the IFS $S = \{\mathbb{R}^2; w_i\}$, $w_i(\mathbf{x}) = \mathbf{B}_i\mathbf{x} + \mathbf{b}_i$, $i = 1, 2, 3, 4$, where:

$$\mathbf{B}_1 = \begin{bmatrix} 0.195 & -0.488 \\ 0.344 & 0.443 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0.722 \\ 0.536 \end{bmatrix};$$

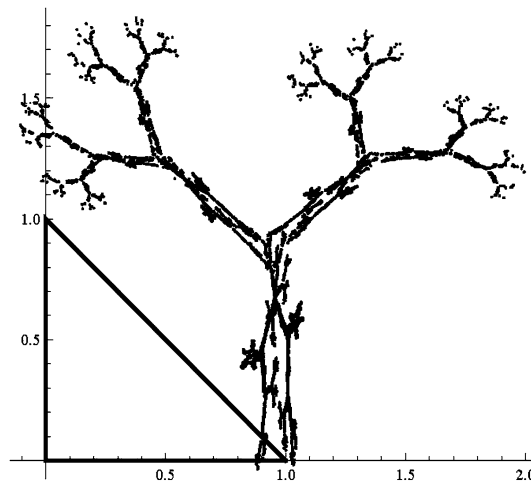


Figure 1 The fractal “Peitgen tree” and the triangle whose vertices form the standard affine basis of $\mathbb{R}^2$, $\mathcal{A} = \{(1,0), (0,1), (0,0)\}$

$$\begin{aligned} \mathbf{B}_2 &= \begin{bmatrix} 0.462 & 0.414 \\ -0.252 & 0.361 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0.538 \\ 1.167 \end{bmatrix}; \\ \mathbf{B}_3 &= \begin{bmatrix} -0.058 & -0.07 \\ 0.453 & -0.111 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1.125 \\ 0.185 \end{bmatrix}; \\ \mathbf{B}_4 &= \begin{bmatrix} -0.045 & 0.091 \\ -0.469 & 0.022 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 0.863 \\ 0.871 \end{bmatrix}. \end{aligned}$$

Find the IFS that generates the translated Peitgen tree for $\mathbf{b} = [1 \ 1]^T$.

Solution. We start by choosing the standard affine basis of $\mathbb{R}^2$, $\mathcal{A} = \{(1,0), (0,1), (0,0)\}$. The Fig. 1 depicts the fractal Peitgen tree and the standard affine basis with connected points.

Translating the affine basis points for $\mathbf{b} = [1 \ 1]^T$, we get the new affine basis, $\mathcal{A}' = \{(2,1), (1,2), (1,1)\}$. We form the matrices

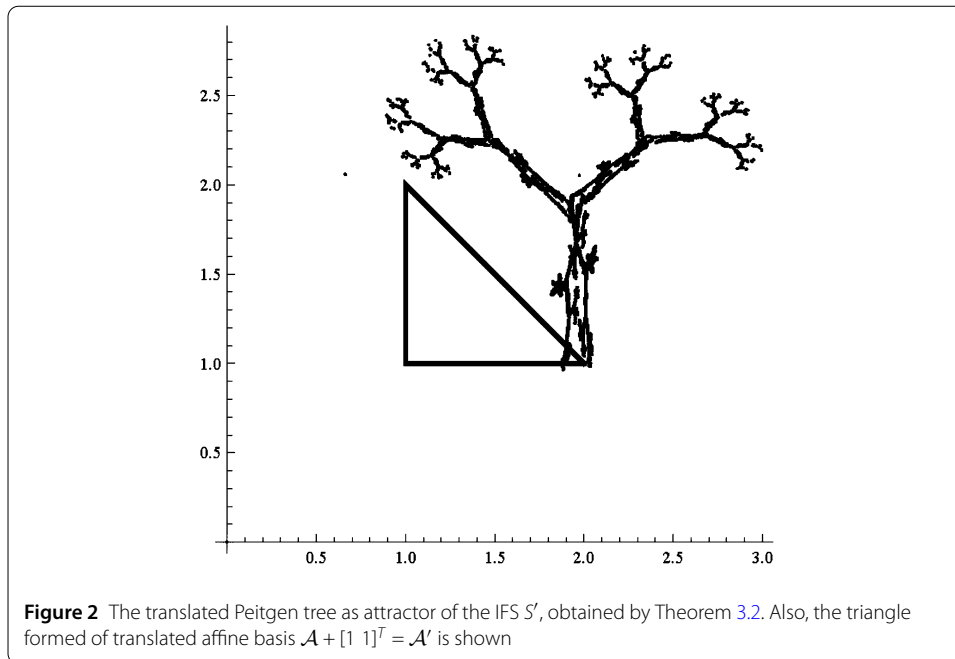
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then, we compute $\mathbf{L}_i$ -s, from

$$\mathbf{L}_i = \mathbf{A}^{-1} \left[\begin{array}{c|c} \mathbf{B}_i & \mathbf{b}_i \\ \hline \mathbf{0}^T & 1 \end{array} \right] \mathbf{A},$$

obtaining

$$\mathbf{L}_1 = \begin{bmatrix} 0.917 & 0.234 & 0.722 \\ 0.88 & 0.979 & 0.536 \\ -0.797 & -0.213 & -0.258 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 1 & 0.952 & 0.538 \\ 0.915 & 1.528 & 1.167 \\ -0.915 & -1.48 & -0.705 \end{bmatrix},$$



$$\mathbf{L}_3 = \begin{bmatrix} 1.067 & 1.055 & 1.125 \\ 0.638 & 0.074 & 0.185 \\ -0.705 & -0.129 & -0.31 \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} 0.818 & 0.954 & 0.863 \\ 0.402 & 0.893 & 0.871 \\ -0.22 & -0.847 & -0.734 \end{bmatrix}.$$

Next, according to

$$\left[\begin{array}{c|c} \mathbf{B}'_i & \mathbf{b}_i \\ \hline \mathbf{0}^T & 1 \end{array} \right] = \mathbf{A}' \mathbf{L}_i (\mathbf{A}')^{-1}.$$

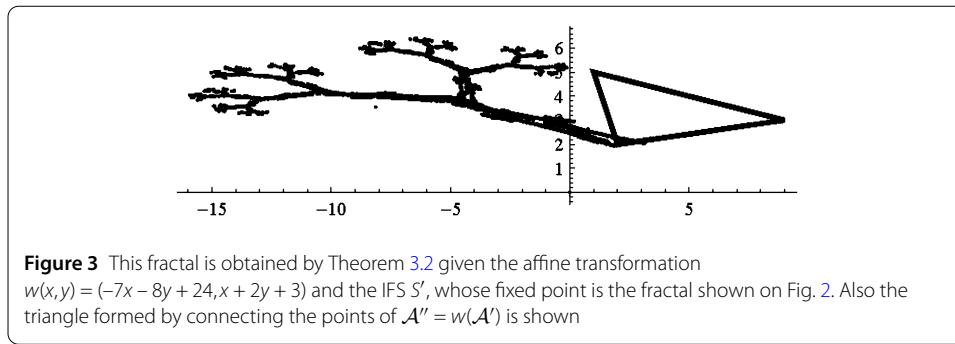
we obtain the matrices and vectors for the target IFS, $S' = \{\mathbf{R}^2; w'_1, w'_2, w'_3, w'_4\}$, $w'_i(\mathbf{x}) = \mathbf{B}'_i \mathbf{x} + \mathbf{b}'_i$, $i = 1, 2, 3, 4$:

$$\begin{aligned} \mathbf{B}'_1 &= \mathbf{B}_1, \quad \mathbf{b}'_1 = \begin{bmatrix} 2.015 \\ 0.749 \end{bmatrix}; & \mathbf{B}'_2 &= \mathbf{B}_2, \quad \mathbf{b}'_2 = \begin{bmatrix} 0.662 \\ 2.058 \end{bmatrix}; \\ \mathbf{B}'_3 &= \mathbf{B}_3, \quad \mathbf{b}'_3 = \begin{bmatrix} 2.253 \\ 0.843 \end{bmatrix}; & \mathbf{B}'_4 &= \mathbf{B}_4, \quad \mathbf{b}'_4 = \begin{bmatrix} 1.817 \\ 2.318 \end{bmatrix}. \end{aligned}$$

The attractor of S' is shown on the Fig. 2.

Example Let's keep S' from the previous example. Find the IFS that generates $w(\text{att}(S'))$, where w is the following affine transformation:

$$w\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -7 & -8 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 24 \\ 3 \end{bmatrix}.$$



Solution. Without loss of generality, we are choosing an initial affine basis to be $\mathcal{A}'$ from the previous example. $\mathcal{A}'' = w(\mathcal{A}')$ will contain the following points:

$$w\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad w\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad w\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 3 \end{bmatrix},$$

i.e. $\mathcal{A}'' = \{(2, 2), (1, 5), (9, 3)\}$.

By Theorem 3.2 we find the IFS $S'' = \{\mathbb{R}^2; w_1'', w_2'', w_3'', w_4''\}$, $w_i''(\mathbf{x}) = \mathbf{B}_i''\mathbf{x} + \mathbf{b}_i''$, $i = 1, 2, 3, 4$, where:

$$\begin{aligned} \mathbf{B}_1'' &= \begin{bmatrix} 0.380 & 1.456 \\ -0.107 & 0.258 \end{bmatrix}, \quad \mathbf{b}_1'' = \begin{bmatrix} -8.132 \\ 3.542 \end{bmatrix}; \\ \mathbf{B}_2'' &= \begin{bmatrix} 0.374 & -1.398 \\ 0.074 & 0.449 \end{bmatrix}, \quad \mathbf{b}_2'' = \begin{bmatrix} -3.271 \\ 2.784 \end{bmatrix}; \\ \mathbf{B}_3'' &= \begin{bmatrix} 0.23 & 1.609 \\ -0.081 & -0.399 \end{bmatrix}, \quad \mathbf{b}_3'' = \begin{bmatrix} -7.250 \\ 4.168 \end{bmatrix}; \\ \mathbf{B}_4'' &= \begin{bmatrix} -0.333 & -1.738 \\ 0.083 & 0.31 \end{bmatrix}, \quad \mathbf{b}_4'' = \begin{bmatrix} 4.199 \\ 2.200 \end{bmatrix}. \end{aligned}$$

The attractor of S'' is shown on the Fig. 3, it is exactly $w(\text{att}(S'))$.

5 Conclusions

In this paper, we addressed a specific inverse problem for iterated function systems: given an IFS with attractor F and an affine transformation w , we constructed a new IFS whose attractor is precisely the affine image $w(F)$. Methods for solving the general inverse problem are typically based on the Collage Theorem and yield only approximations of the target set, whereas our method provides an exact solution to the stated inverse problem. The construction of the new IFS is carried out using barycentric (affine) coordinates, whose key properties are formulated and proved prior to establishing the main result presented in Theorem 3.2. Along with the IFS, the formula for the corresponding Hutchinson operator is stated. To make the method reproducible and transparent, the pseudocode and detailed worked examples are provided.

Future work may extend this method for local control of fractal attractors or for solving other instances of the inverse problem involving different transformations. Beyond

its theoretical significance, the approach presented here contributes to the broader understanding of fractal transformations and opens possibilities for a wide range of applications. Exact control over affine transformations of fractal attractors is valuable in computer graphics, where fractal geometry is often used for realistic modeling of natural objects and textures. The method also has potential applications in image processing, particularly in areas related to fractal image compression and geometric modeling. Moreover, since the algorithm is constructive and numerically straightforward, it is well suited for software implementations that support interactive manipulation of fractals in both two and three dimensions.

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Data availability

No datasets were generated or analysed during the current study.

Declarations**Usage of generative AI and AI-assisted technologies in the writing process**

During the preparation of this work, the author used Grammarly and ChatGPT to correct grammatical and syntactical errors, improving the study's correctness and readability. After using these tools/services, the author reviewed and edited the content as needed and takes full responsibility for the publication's content.

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